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An approach to anomalies in M-theory via KSpin

Hisham Sati*

Department of Mathematics, Yale University, New Haven, CT 06520, USA

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Abstract

The M-theory field strength and its dual, given by the integral lift of the left-hand side of the equation of motion, both satisfy certain cohomological properties. We study the combined fields and observe that the multiplicative structure on the product of the corresponding degree four and degree eight cohomology fits into that given by Spin K-theory. This explains some earlier results and leads naturally to the use of Spin characteristic classes. We reinterpret the one-loop term in terms of such classes and we show that it is a homotopy invariant. We argue that the various anomalies have natural interpretations within Spin K-theory. In the process, mod 3 reductions play a special role.

Keywords: Topological anomalies in M-theory; K-theory and generalized cohomology; Spin bundles; Characteristic classes

1. Introduction

The non-gravitational fields in M-theory and string theory play a major role in characterizing the topology and the global aspects of these theories. Such fields take continuous real or complex values in the classical supergravity limit and get quantized, so that a priori they take values in $\mathbb{Z}$, in the quantum regime. The fields take values in cohomology of the space X, and so classically are in $H^*(X, \mathbb{R})$ and quantum-mechanically in $H^*(X, \mathbb{Z})$. An important difference between the two cases is the presence of torsion in the latter case and that does not exist in the former. It is in fact this feature that gives the subtle distinction between (generalized) cohomology theories.

Both G_4 and its 'dual' – let us call it G_8 for now – involve shifts in the Pontrjagin classes. The M-theory degree four field G_4 defined on an eleven-dimensional space Y^{11} is not an integral class but satisfies the shifted integrality condition [1] $G_4 - p_1/4 \in H^4(Y^{11}, \mathbb{Z})$, where p_1 is the first Pontrjagin class of the tangent bundle TY^{11}. This is written as [1]

$$G_4 - \lambda/2 \in H^4(Y^{11}, \mathbb{Z}), \tag{1.1}$$

where λ is equal to half the Pontrjagin class of the eleven-dimensional space Y^{11}. Compared to ten-dimensional string theory, described by K-theory, at the level of partition functions, torsion fields play a major role [2]. In particular they

* Tel.: +1 203 432 2876.
E-mail address: hisham.sati@yale.edu.

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lead to an anomaly for the partition function. This is canceled in [3] by declaring spacetime to be oriented with respect to generalized cohomology theories beyond K-theory.

In addition to this field that appears in the eleven-dimensional supergravity multiplet, there is also the dual field whose class is considered in [4–7] and has a quantization condition of its own. This is the class given by the integral lift of the right-hand side of the equation of motion for G_4 [4]. G_8 is built out of a quadratic term in G_4 plus the one-loop term, which is a polynomial expression in the Pontrjagin classes p_1 and p_2. In [6,7], a distinction is made between two fields that can be dual to G_4: the actual Hodge dual $*G_4$ and the class Θ defined in [4].

In this note, we investigate the multiplicative structure on the product of the cohomology of degrees four and eight. In particular we will show that the quadratic refinement defined in [4] is encoded in the multiplicative structure in the K-theory for Spin bundles. This will motivate us to propose that the Spin characteristic classes are the natural setting for the above shifts. This gives an insight into the relation between G_4 and its 'dual'. We then make connection to the classes proposed in [5]. The calculation of the path integral involves exponentiating the action time $2\pi i$. The requirement that the partition function is well-defined imposes integrality properties on the topological terms of the action. One such term is the one-loop term (Eq. (2.4)), whose integrality was established in [1] using congruence from index theory. This term takes an interesting form when written in terms of the Spin characteristic classes. In fact, it turns out to be essentially given by the second Spin class, up to an interesting factor of 24 which reminds us of other occurrences of such a factor. As a warm up to discussing the mod p reduction of the fields, we show that the one-loop term is a homotopy invariant. The two facts strongly suggest that this term should have a deep homotopy-theoretic meaning.

The observation that the quadratic refinement is given by the natural multiplication on the image of the Chern character motivates us to seek more connections with KSpin. To make such connections we study the mod 3 reductions of the fields. The anomalies in M-theory and type IIA string theory are encoded as conditions on the natural bundles and the aim here is to argue for a unified approach. We provide evidence for this from the quadratic structure as well as from the form of the anomalies themselves. This, however, leaves many interesting and subtle questions open, such as accounting for the precise denominator factors, most importantly the factors $\frac{1}{2}$ and $\frac{1}{24}$. Nevertheless, one observation is the connection between $p = 3$ and M-theory and between $p = 2$ and string theory, which provides more systematic evidence for observations in our previous work [7]. Another theme is the mod 24 quantization. What we see is that this approach seems to treat in a unified way the anomalies in the membrane theory, in type IIA string theory, in the five-brane theory, and in M-theory. In terms of classes, roughly, the M2-brane corresponds to the first Spin class and the M5-brane [8,9] corresponds to the second Spin class.

Anomalies generally involve Spin bundles and so it is only natural to study them within K-theory of such bundles. How is Spin K-theory related to more well-known K-theories? Given a topological space X, let $\widetilde{KO}(X)$ be the reduced *KO* group for X and let

$$W : \widetilde{KO}(X) \longrightarrow H^1(X; \mathbb{Z}_2) \times H^2(X; \mathbb{Z}_2) \tag{1.2}$$

be the map $W(\xi) = (w_1(\xi), w_2(\xi))$, where $w_i(\xi)$ denotes the ith Stiefel–Whitney class of $\xi \in \widetilde{KO}(X)$. There is a group structure on $H^1(X; \mathbb{Z}_2) \times H^2(X; \mathbb{Z}_2)$ making W a homomorphism, i.e. a map that preserves the group structure. Starting with a real unoriented bundle ξ, the condition $w_1(\xi) = 0$ turns ξ into an oriented bundle, and the condition $w_2(\xi) = 0$ further makes ξ a Spin bundle. Obviously then, a real O-bundle becomes a Spin bundle when $W = 0$, and so the kernel of W is the reduced group (see Section 7) $\widetilde{K\mathrm{Spin}}(X)$. Thus W fits into the exact sequence [10]

$$0 \longrightarrow \widetilde{K\mathrm{Spin}}(X) = \ker W \longrightarrow \widetilde{KO}(X) \xrightarrow{W} H^1(X; \mathbb{Z}_2) \times H^2(X; \mathbb{Z}_2). \tag{1.3}$$

We do not consider specific examples since KSpin of many classes of interesting spaces are already tabulated in [10].

We say that $x \in H^*(X; \mathbb{Z})$ is an element of order r ($r = 2, 3, 4, \ldots$) if and only if $x \neq 0$ and r is the least positive integer such that $rx \neq 0$ (if it exists). The reduction mod k induces the mapping $\rho_k : H^*(X; \mathbb{Z}) \to H^*(X; \mathbb{Z}_k)$. For more background on cohomology operations, see e.g. [11].

2. The one-loop term via Spin characteristic classes

Recall that characteristic classes on a space X are obtained by pulling back to the space X the universal classes from the cohomology ring of the corresponding universal space. For oriented vector bundles, the relevant group is SO

with classifying space BSO. Rationally, the cohomology ring $H^*(BSO;\mathbb{Q})$ is a polynomial ring over $\mathbb{Q}$ generated by the universal Pontrjagin classes $p_i \in H^{4i}(BSO;\mathbb{Q})$.

As is the case for any G-bundle, Spin bundles have a classifying space, which is BSpin, and the corresponding characteristic classes are obtained by pulling back from that space. More precisely, the Spin characteristic classes can be defined for the stable class of a Spin bundle ξ over a topological space X, in our case an eight-, eleven- or twelve-dimensional space, by $Q_i(\xi) = \iota^* Q_i \in H^4(X;\mathbb{Z})$, where $\iota : X \longrightarrow B\mathrm{Spin}$ is the classifying map, in the stable range, for the bundle ξ. The corresponding Q_i are cohomology classes $Q_i \in H^{4i}(B\mathrm{Spin};\mathbb{Z})$, for $i = 1, 2, \ldots$.

The Spin cohomology ring with coefficients in $\mathbb{Z}_2$ is generated by the mod 2 Stiefel–Whitney classes of certain degrees [12]. What we are interested in is integral coefficients, in which case

$$H^*(B\mathrm{Spin};\mathbb{Z}) = \mathbb{Z}[Q_1, Q_2, \ldots] \oplus \gamma, \tag{2.1}$$

with γ a 2-torsion factor, i.e. $2\gamma = 0$ [13]. The two degrees relevant to our discussion are

$$\begin{aligned} &H^4(B\mathrm{Spin};\mathbb{Z}) \cong \mathbb{Z} \quad \text{with generator } Q_1 \\ &H^8(B\mathrm{Spin};\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z} \quad \text{with generators } Q_1^2, Q_2, \end{aligned} \tag{2.2}$$

where Q_1 and Q_2 are determined by their relation to the Pontrjagin classes

$$\begin{aligned} p_1 &= 2Q_1 \\ p_2 &= Q_1^2 + 2Q_2. \end{aligned} \tag{2.3}$$

Obviously, when inverting is possible, the generators are given by $Q_1 = p_1/2$ and $Q_2 = \frac{1}{2}p_2 - \frac{1}{2}(p_1/2)^2$.

We now make the first use of the Spin classes. In particular we use them to write the one-loop polynomial I_8 in a suggestive way, and we then make connection to the classes proposed in [5]. The one-loop polynomial of some tangent bundle T is given in terms of the Pontrjagin classes [14]

$$I_8 = \frac{p_2(T) - (p_1(T)/2)^2}{48}, \tag{2.4}$$

where $p_1/2$ is usually denoted λ, and represents the string class. In an earlier work [5] we observed that I_8 can be written in a way that suggests its interpretation as a Chern character[1] upon using the class λ – which we called λ_1 in [5] – and another class, which we defined as $\lambda_2 = p_2/2$, were used. This led to the expression

$$I_8 = \frac{\lambda_2 - \frac{1}{2}\lambda_1^2}{24}. \tag{2.5}$$

Now we proceed to write I_8 in terms of the Spin classes Q_1 and Q_2 and compare the result with (2.5). For that we simply substitute (2.3) to get

$$I_8 = \frac{Q_2}{24}. \tag{2.6}$$

First, note that this expression is written entirely in terms of the second Spin characteristic class Q_2 as the first one, Q_1, canceled out. The relation to the classes in [5] is now obvious. The class λ_1 is exactly Q_1, whose values is half the first Pontrjagin class. The degree eight class λ_2 is then equal to Q_2 once Q_1 vanishes. This has a nice interpretation. Since we are viewing the classes Q_i as obstructions, then it makes sense to be able to talk about the second obstruction only after the first obstruction is absent. This then gives the desired structure to the observations and proposal in [5,6] on the Spin part of the polynomials.

3. Topological and homotopy invariance

In this section we investigate whether the classes used in [5] and the one-loop term (2.4) are topological invariant and/or homotopy invariant. Homotopy invariance means dependence only on the homotopy type of the manifold,

[1] We will come back to the character interpretation in Section 7.

and independence of the differentiable structure. Topological invariance, on the other hand, is the requirement of independence on the *choice* of a differentiable structure. In both the cases, the statements depend on the coefficient ring over which the Pontrjagin classes are taken.

3.1. Homotopy invariance of Pontrjagin classes

The homotopy invariance of the rational Pontrjagin classes p_k depends on whether one is considering stable or unstable bundles. For stable universal vector bundles, $p_k \in H^{4k}(BO, \mathbb{Q})$ are not homotopy invariant for $k \geq 1$, but for nonstable vector bundles $p_k \in H^{4k}(BO[2k], \mathbb{Q})$ are homotopy invariant [15]. The situation for the integral Pontrjagin classes modulo[2] q is as follows. For $q = 2$, p_k mod $2 = w_{2k}^2$, and since the Pontrjagin classes are homotopy invariant, this implies that p_k mod 2 are homotopy invariant. We deduce from this that the classes $p_i/2$ used in [5] are homotopy invariant. The integral Pontrjagin classes p_k modulo q, where q is an odd prime, are homotopy invariant only if $q = 3$. A classic result of Wu that p_k mod 3 are the Wu classes U_3^k, which are defined in terms of the Steenrod reduced powers (see Section 4) implies that they are homotopy invariant. Thus integral p_k mod q are not homotopy invariant for any other $q \neq 3$ [16].

3.2. Topological invariance of Pontrjagin classes

For a topological manifold M (for us, Z^{12}, Y^{11}, X^{10}, or M^8), let Σ_1 and Σ_2 be two different smooth structures and let TM_{Σ_1} and TM_{Σ_2} be the corresponding tangent bundles. Associate the kth Pontrjagin classes $p_k(TM_{\Sigma_1})$ and $p_k(TM_{\Sigma_2})$ in $H^{4k}(M, \Lambda)$. The question is whether or not $p_k(TM_{\Sigma_1}) = p_k(TM_{\Sigma_2})$. It turns out that the answer depends on the coefficient ring Λ. For $\Lambda = \mathbb{Q}$, it is a classic result of Novikov that the rational Pontrjagin classes are topological invariants. However, this is *not* the case for the integral case $\Lambda = \mathbb{Z}$. What about $\Lambda = \mathbb{Z}_q$, the ring of integers q, where q is any prime? In this case, as mentioned above, p_k mod 3 are the Wu classes U_3^k, which are defined in terms of the Steenrod reduced powers (see Section 4) and hence are topological invariants. This has been extended to $q = 5$ in [15]. Thus, for every $k \geq 1$, p_k mod q are topological invariants for $q = 3$ and 5. However, this breaks down at $q = 7$ as then p_2 mod 7 is not topological invariant [17].

Since the integral Pontrjagin classes are not topological invariant, one can ask: what are the multiples of the integral p_k's that are topological invariant? The smallest possible integer n_k such that $n_k p_k$ is a topological invariant is given by $n_1 = 1$ and $n_2 = 7$ [18].

3.3. Consequences for the one-loop term

We would like to investigate the invariance of the one-loop term (2.4) in the context of the above discussion. The one-loop term is an example of a Ponrjagin number, i.e. a polynomial of a given degree in the Pontrjagin classes. It is known that at the rational level, the only rational linear combination in the Ponrjagin classes that is homotopy invariant is, up to a rational linear multiple, the Hirzebruch L-polynomial [19] that appears in the signature theorem. However, the one-loop term is not quite equal to L_2 (see (6.2) for the corresponding expression) and thus the polynomial (2.4) cannot be homotopy invariant at the rational level. Thus we are forced to study the expression modulo primes.

In addition to homotopy invariance of the Pontrjagin classes mod 3, there is an additional result [20] that p_k mod 2^3 are also homotopy invariant. Thus p_k mod 24 are homotopy invariant. In particular this means that p_2 mod 24 is homotopy invariant. We are still short by a factor of 2 to get the first term in (2.4). Let us look at the analogous situation for p_1. In that case, the fact that $p_1(\xi) \equiv w_2(\xi)^2$ mod 2 implied the fact that p_1 is even when the bundle ξ is Spin, because then $w_2(\xi) = 0$. Combining the two results one has that the first Pontrjagin class of a Spin bundle is a homotopy invariant mod 24. Now let us see what can be said about p_2. Here note that $p_2(\xi) \equiv w_4(\xi)^2$ mod 2, so that we do get the evenness of p_2 provided that we have the condition $w_4(\xi) = 0$, the higher degree analog that replaces the spin condition. Note that this is the obstruction to orientation with respect to the real version $EO\langle 2\rangle$ of Landweber elliptic cohomology with two generators which appears in the study of the partition functions [3,21,22]. Given this condition, we are then able to define $p_2/2$ as in [5]. Going back to the one-loop term, we have so far that the first term in (2.4) is homotopy invariant.

[2] We use q instead of p to denote a prime, so as not to confuse with the several variations on p used in this note.

What about the second term in (2.4)? We consider p_1^2. Since p_k mod 3 are homotopy invariant then so is p_k^m mod 3. In particular, then, p_1^2 mod 3 are homotopy invariant. For Spin bundles p_1 is even so then $(\frac{1}{2}p_1)^2$ mod 3 is a homotopy invariant and so p_1^2 mod 12 is a homotopy invariant. On the other hand, from [20], p_1 mod 2^3 is a homotopy invariant. Combining the two results implies that p_1^2 mod 96 is a homotopy invariant. Therefore, the one-loop term is a homotopy invariant. In fact, as we have just seen, we have more: each of the two terms separately is homotopy invariant.

4. The multiplicative structure on the cohomology ring

In addition to the usual cohomology ring $H^*(X;\mathbb{Z}_q)$ which is the direct sum of the elements in the individual degrees depending on grading, one can also form the direct product $H^{**}(X;\mathbb{Z}_q)$ of the cohomology groups $H^i(X;\mathbb{Z}_q)$ for $i = 0, 1, 2, \ldots$. In this way, the direct sum $H^*(X;\mathbb{Z}_q)$ can be thought of as being included inside $H^{**}(X;\mathbb{Z}_q)$. The ring structure on both H^* and H^{**} is given by the cup-product operation. Inside the ring $H^{**}(X;\mathbb{Z}_q)$ one can also talk about inverting elements x, which is possible when the zeroth component is nonzero in $H^0(X;\mathbb{Z}_q)$.

One can form the total Steenrod reduced power operation $P = P^0 + P^1 + P^2 + \cdots$ which acts as an automorphism of rings $H^{**}(X;\mathbb{Z}_q) \longrightarrow H^{**}(X;\mathbb{Z}_q)$, and is the identity on $H^*(X;\mathbb{Z}_q)$. The cohomology ring $H^{**}(X)$ is graded and decomposes as $H^{**}(X) = H^{\text{even}}(X) + H^{\text{odd}}(X)$ where $H^{\text{even}}(X) = \prod_{m=0}^{\infty} H^{2m}$ and $H^{\text{odd}}(X) = \prod_{m=0}^{\infty} H^{2m+1}$ are the cohomology groups in even and odd degrees, respectively. For coefficients $\mathbb{Z}_q$, one has the characteristic classes $\overline{p}_i$ as the mod q reduction of the Pontrjagin classes p_i generating the ring $H^{**}(BO;\mathbb{Z}_q) = \mathbb{Z}_q[[\overline{p}_1, \overline{p}_2, \ldots]]$.

As in the case for the mod 2 classes, i.e. the Stiefel–Whitney classes, one can form the Wu classes, and the construction is analogous. We now have an orientation so we work with BSO rather than BO. By using the 'inverse' P^{-1} of the operation P, one can define

$$U(P) = P^{-1}\phi^{-1}P\phi(1) \in H^{**}(BSO;\mathbb{Z}_q), \tag{4.1}$$

where ϕ is the extension to $H^{**}(BSO;\mathbb{Z}_q)$ of the Thom isomorphism $\phi : H^*(BSO;\mathbb{Z}_q) \longrightarrow H^*(MSO;\mathbb{Z}_q)$, and 1 is the unit in $H^0(X;\mathbb{Z}_q)$.

The above is indeed analogous to the more familiar result for the mod 2 Wu class that uses the total Steenrod operation Sq,

$$v(Sq) = Sq^{-1}\phi^{-1}Sq\phi(1) \in H^{**}(BO;\mathbb{Z}_q). \tag{4.2}$$

Applying Sq to (4.2) gives the class $Sq\,v(Sq) \in H^{**}(BO;\mathbb{Z}_q)$ as the direct product of the universal Stiefel–Whitney classes. Likewise, applying P to (4.1) gives the classes $q_i = (PU(P))_i$ as the direct product of the universal mod 3 classes. The classic results of Wu imply that the classes q_i are *oriented* homotopy invariants and the Stiefel–Whitney classes are homotopy invariants.

The Wu classes can be written in terms of multiples of the Hirzebruch L-polynomials [23,24]. For every prime q certain polynomials (with respect to the cup-product) in the Pontrjagin classes p_i reduced mod q are topological invariants (mod q). For $q = 2$ of course one has the Stiefel–Whitney classes. Since $p_i = w_{2i}^2$ (mod 2) then p_i reduced mod 2 is invariant. For q an odd prime, the Steenrod powers P_q^r lead to certain polynomials $U_q^r \in H^{2r(q-1)}(M^m;\mathbb{Z}_q)$ in the Ponrjagin classes which are topologically invariant, and which are characterized by the property

$$P_q^r(v) = U_q^r(v) \quad \text{for all } v \in H^{m-2r(q-1)}(M^m;\mathbb{Z}_q). \tag{4.3}$$

As mentioned before, these can be written in terms of the Hirzebruch L-polynomials as

$$U_q^r = q^r L_{\frac{1}{2}r(q-1)}(p_1, p_2, \ldots) \ (\text{mod } q). \tag{4.4}$$

Thus (for M^8) the first Steenrod power at the prime $p = 3$ is

$$\begin{aligned} U_3^1 &= 3L_1 \bmod 3 \\ &= 3\frac{p_1}{3} \bmod 3 \\ &= p_1 \bmod 3. \end{aligned} \tag{4.5}$$

5. Action of the Steenrod reduced powers

Since the Steenrod reduced power operation P_q^r raises the cohomology degree by $2r(q-1)$, we see that the highest prime that keeps us within dimension twelve is $q = 5$. The possible stable operations in that range are[3]

(i) $q = 2$: Sq^i for $i \leq 6$,
(ii) $q = 3$: P^1, βP^1, P^2, βP^2,
(iii) $q = 5$: P^1, βP^1.

We are further interested only in degree four classes, that we would like to either square or cube, and in degree seven and degree eight classes whose degree we raise only up to a maximum of twelve.

Let us start with the degree four class. Note that the βP_q^i are of odd dimension and thus are not useful in this case. They, however are useful in type II string theory (see [25]) and later for the discussion of G_7. Thus, we are left with only Sq^4 and P_3^1, which square a degree four class, and with P_3^2 and P_5^1, which cube a degree four class. So we see just from this dimensional analysis that the first pair makes up the candidates in 8 dimensions, whereas the second two are the candidates in 12 dimensions. Of course this analysis is only to motivate the discussion and later we will resort to more precise arguments that come from making the connection to Spin K-theory.

The Adem relation in the mod q Steenrod algebra for the Steenrod powers involving P_q^1 is

$$P_q^1 P_q^{2^k-1} = 2^k P_q^{2^k}. \tag{5.1}$$

Then if the dimension of the generator x is 2^{k+1}, the Adem relation on x gives

$$x^n = \frac{1}{2^k} P_q^1 P_q^{2^k-1} x \ (\mathrm{mod}\ q) \tag{5.2}$$

where x^n is the cup-product n-power of x, $\underbrace{x \cup x \cdots \cup x}_{n}$. From this one can easily get a restriction on the degree in order to have a nonzero cube. For $q = 3$,

$$x \cup x \cup x = (-1)^k P_3^1 P_3^{2^k-1} x \ (\mathrm{mod}\ 3), \tag{5.3}$$

and since the dimensions of $P_3^{2^k-1}x$ is $3.2^{k+1} - 4$, we see that the cube $x \cup x \cup x$ is zero (mod 3) unless $3.2^{k+1} - 4$ is a multiple of 2^k. This happens only for $k = 0$ and $k = 1$,

(i) $k = 0$: $\dim x = 2$, $x_2 \cup x_2 \cup x_2 = P_3^1 x_2$ (mod 3)
(ii) $k = 1$: $\dim x = 4$, $x_4 \cup x_4 \cup x_4 = \frac{1}{2} P_3^1 P_3^1 x_4$ (mod 3) $= P_3^2 x_4$ (mod 3).

Let us consider the latter case, where $x_4 \in H^4(X; \mathbb{Z})$ is an integral generator. The Adem relation $P^1 P^1 = 2P^2$ for a general prime q implies that

$$P^1 \overline{x}_4 = \pm 2 \overline{x}_4^{\frac{(q+1)}{2}} \tag{5.4}$$

in $H^*(X; \mathbb{Z}_p)$, where $\overline{x}_4$ is the mod q reduction of the integral generator x_4. Therefore, we have

(i) $q = 3$: $P_3^1 \overline{x}_4 = \pm 2 \overline{x}_4^2$ with $\overline{x}_4 = \rho_3(x_4)$,
(ii) $q = 5$: $P_5^1 \overline{x}_4 = \pm 2 \overline{x}_4^3$ with $\overline{x}_4 = \rho_5(x_4)$, where ρ_q, again, denotes reduction modulo q.

6. Modulo 3 reductions of the fields

In this section we consider the mod 3 reduction of the fields and we consider the possible actions of the admissible cohomology operations on them.

[3] In this list we omit the subscript q as it is obvious.

6.1. The degree four field

The first Steenrod reduced power operation for $\mathbb{Z}_3$ cohomology is P_3^1, which takes elements in $H^k(X;\mathbb{Z}_3)$ into elements of $H^{k+4}(X;\mathbb{Z}_3)$. We consider the mod 3 reduction $x_4=\rho_3(G_4)$ of the M-theory field G_4. We know from Ref. [1] that G_4 extends to the twelve-dimensional bounding theory on Z^{12}, i.e. such that the eleven manifold Y^{11} is ∂Z^{12}. In this case, in addition to the first Steenrod reduced power P_3^1 at $p=3$ (outlined above and will be discussed further in Section 6.3), we can also consider the second operation P_3^2, which raises the cohomology degree by eight. Thus we have

$$P_3^2 x_4 \in H^{12}(Z^{12},\mathbb{Z}_3), \tag{6.1}$$

which is equal to $U_3^2 x_4$, where now

$$\begin{aligned} U_3^2 &= 3^2 L_2 \bmod 3 \\ &= 3^2 \frac{1}{45}(7p_2^2 - p_1^2) \bmod 3 \\ &= \frac{7p_2 - p_1^2}{5} \bmod 3. \end{aligned} \tag{6.2}$$

Thus, the action of P_3^2 on the mod 3 reduction of G_4 is

$$P_3^2 x_4 = \rho_3\left(\frac{7p_2 - p_1^2}{5}\right) x_4. \tag{6.3}$$

Since G_4 also involves a gravitational shift that involves p_1, we also mention the action of power operations on the first Pontrjagin class. The mod 3 reduction of the Pontrjagin class $\rho_3(p_1(\xi))$ is an element in $H^4(X;\mathbb{Z}_3)$, given by the Wu class $U_3^1(\xi)$. Thus we can have an action of P_3^1, and the result is

$$\begin{aligned} P_3^1\rho_3(p_1(\xi)) &= P_3^1 U_3^1(\xi) \\ &= \rho_3\left(2p_2(\xi) - p_1^2(\xi)\right). \end{aligned} \tag{6.4}$$

6.2. The degree seven dual field G_7

We are interested in the action of cohomology operations (at $q=3$) on the fields (reduced modulo 3). Since the smallest dimension for such an operation is four, this means that we cannot consider the dual degree eight class Θ (in the notation of [4]) without going beyond eleven dimensions. We can, however, consider the differential form $G_7 = *_{11} G_4$, on which we perform the mod 3 reduction after lifting to an integral class. Let us call the resulting class $x_7 \in H^7(Y^{11},\mathbb{Z}_3)$. In this case the first Steenrod reduced power P_3^1 at $q=3$ acts on x_7 to give a top class

$$P_3^1 x_7 \in H^{11}(Y^{11},\mathbb{Z}_3). \tag{6.5}$$

This top-dimensional element is characterized by the Poincaré duality theorem[4] and is given by the class $U_3^1 x_7$. The element U_q^r is given by (4.4). Adapting to our situation, with $p=3$, $r=1$, we have the Wu class U_3^1 (Eq. (4.5)). Therefore, the action of P_3^1 on the mod 3 reduction of G_7 is given by

$$P_3^1 x_7 = \rho_3(p_1)\cup x_7 = U_3^1 \cup x_7. \tag{6.6}$$

[4] Note that (6.5) is a top class in $\mathbb{Z}_3$. Such situations may occur (at least for homology) when the space is not a closed manifold but rather a manifold with multiple boundary components together with an identification of these components. A standard class of examples is the so-called $\mathbb{Z}_k$- (or $\mathbb{Z}/k$-)manifolds of Sullivan.

6.3. The degree eight 'dual' field Θ

Here we would like to act by cohomology operations on the mod 3 reduction $\rho_3(\Theta) = y_8$ of the class Θ. Assuming that the class extends to twelve dimensions, we can consider

$$P_3^1 y_8 \in H^{12}(Z^{12}, \mathbb{Z}_3). \tag{6.7}$$

As in the case for G_7 this is equal to $U_3^1 y_8$, so that

$$P_3^1 \rho_3(\Theta) = \rho_3(p_1)\rho_3(\Theta), \tag{6.8}$$

which is analogous to (6.6).

Next we show that the degree eight class $\Theta(\rho_3(a))$ corresponding to the mod 3 reduction can be written as a cohomology operation. We use (5.4) and the additivity of the mod k reduction, i.e. $\rho_k(a+b) = \rho_k(a) + \rho_k(b)$, to calculate for G_4 reduced mod 3, $\overline{G_4}$, the following[5]

$$\begin{aligned}
\frac{1}{2}\left[\frac{1}{2}P_3^1\overline{G}_4 + \overline{G}_4 \cup \overline{G}_4\right] &= \frac{1}{2}\left[\frac{1}{2}P_3^1\left(\rho_3(a) - \rho_3(\lambda/2)\right) + \left(\rho_3(a) - \rho_3(\lambda/2)\right) \cup \left(\rho_3(a) - \rho_3(\lambda/2)\right)\right] \\
&= \rho_3(a) \cup \rho_3(a) - \rho_3(\lambda/2) \cup \rho_3(\lambda/2) + \rho_3(a) \cup \rho_3(a) \\
&\quad + \rho_3(\lambda/2) \cup \rho_3(\lambda/2) - \rho_3(a) \cup \rho_3(\lambda) \\
&= \left[2\rho_3(a) \cup \rho_3(a) - \rho_3(a) \cup \rho_3(\lambda)\right] \\
&= \Theta\left(\rho_3(a)\right),
\end{aligned} \tag{6.9}$$

the DFM class with the degree eight term set to zero. The full result will involve the reduction of I_8. The division by two on the left-hand side is harmless since we are reducing modulo 3. This may be thought of as mod 3 analog in M-theory of the mod 2 expression in type II string theory, namely the Freed–Witten anomaly cancelation formula for D-branes [26] $(Sq^3 + H_3 \cup)F = 0$, since the class Θ measures the anomaly of the M-branes [4].

7. The quadratic refinement and Spin K-theory

In this section we will show that the multiplicative structure on the degree four and degree eight cohomology encodes the quadratic refinement law of [4] for the eight-form in M-theory, the refinement being given by the cup-product of two four-forms from G_4. We will see that this is reflected in the addition on the target (Eq. (7.8)).

The degree eight class in M-theory is given by the integral lift of the (negative of the) right-hand side of the equation of motion for G_4, which is

$$d * G_4 = -\frac{1}{2}G_4 \wedge G_4 + I_8, \tag{7.1}$$

so that the degree eight class $\Theta(a)$, defined in [4], is

$$\Theta(a) = \left[\frac{1}{2}G_4 \wedge G_4 - I_8\right], \tag{7.2}$$

whose expression in terms of integral classes a and λ reads

$$\Theta(a) = \frac{1}{2}a(a-\lambda) + 30\widehat{A}_8. \tag{7.3}$$

Among the properties of this class proved in [4] is that it is a quadratic refinement of the cup-product of two degree four classes a_1 and a_2

$$\Theta(a_1 + a_2) + \Theta(0) = \Theta(a_1) + \Theta(a_2) + a_1 \cup a_2. \tag{7.4}$$

5 Here we assume that $\overline{G}_4$ is in cohomology. This would come from assuming that both factors in the shifted quantization condition [1] to be in integral cohomology, an so the mod q reduction is in mod q cohomology.

We would like to look at this from the point of view of the structure on the product of the cohomology groups $H^4(\,;\mathbb{Z}) \times H^8(\,;\mathbb{Z})$. For this we consider the two classes a and $\Theta(a)$ as a pair $(a, \Theta(a))$ in $H^4(\,;\mathbb{Z}) \times H^8(\,;\mathbb{Z})$. Then the linearity of the addition of the degree four classes a and the quadratic refinement property (7.4) of $\Theta(a)$ can both be written in one expression in the product $H^4(\,;\mathbb{Z}) \times H^8(\,;\mathbb{Z})$, which makes use of the ring structure, namely

$$(a_1, \Theta(a_1)) + (a_2, \Theta(a_2)) = (a_1 + a_2, \Theta(a_1) + \Theta(a_2) + a_1 \cup a_2)\,. \tag{7.5}$$

The second entry on the right-hand side is just $\Theta(a_1 + a_2) - \Theta(0)$, and so it encodes the property (7.4).

We can define the shifted class $\Theta^0(a)$ as the difference $\Theta(a) - \Theta(0)$, so that (7.5) is replaced by

$$\left(a_1, \Theta^0(a_1)\right) + \left(a_2, \Theta^0(a_2)\right) = \left(a_1 + a_2, \Theta^0(a_1 + a_2)\right), \tag{7.6}$$

corresponding to the special case

$$\Theta^0(a_1 + a_2) = \Theta^0(a_1) + \Theta^0(a_2) + a_1 \cup a_2. \tag{7.7}$$

This is then just a realization of the multiplication law on $H^4(\,;\mathbb{Z}) \times H^8(\,;\mathbb{Z})$ which, for (a, b) in the product group, is

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2 + a_1 \cup a_2). \tag{7.8}$$

Note that in order to get this law we had to use the modified eight-class $\Theta^0(a)$, or alternatively discard $\Theta(0) = 30\widehat{A}_8$.[6] From the quadratic refinement law, [4] noted that this term can at most be 2-torsion.

We now make the connection to Spin K-theory. Similarly to the case of other kinds of bundles, e.g. complex or real, one can get a Grothendieck group of isomorphism classes of Spin bundles up to equivalence. The reduced KSpin group of a topological space can be defined as $\widetilde{K\mathrm{Spin}}(X) = [X, B\mathrm{Spin}]$. For the case of BSpin, we will be interested in relating Spin K-theory to cohomology of degrees 4 and 8. Such a homomorphism of abelian groups

$$Q_X : \widetilde{K\mathrm{Spin}}(X) \to H^4(X;\mathbb{Z}) \times H^8(X;\mathbb{Z}) \tag{7.9}$$

is defined by [10] $Q_X\,(Q_1(\xi), Q_2(\xi))$ for $\xi \in \widetilde{K\mathrm{Spin}}(X)$. We see that this is the Spin analog of (1.2). For two bundles ξ and γ in $\widetilde{K\mathrm{Spin}}(X)$, and for $k \le 3$,

$$Q_k(\xi \oplus \gamma) = \sum_{i+j=k} Q_i(\xi) \cup Q_j(\gamma). \tag{7.10}$$

We also see that the map (7.9) is essentially our 'gravitational' Chern character in [5]. The fact that this relation only works for $k \le 3$ is in accord with the observation that the expressions in [5] also only work for that range.[7] The addition on the target is given precisely by (7.8) for $(a, b) \in H^4(X;\mathbb{Z}) \times H^8(X;\mathbb{Z})$ [13].

The quantization condition on G_4 [1] (see the introduction) involves an integral class coming from the E_8 bundle. How does this E_8 part fit into the above discussion? Since $H^8(E_8) = 0$, then any degree eight class would have to come from the only class of lower degree, namely the degree four class. The only possibility is squaring. Indeed, using Chern–Weil representatives, $\mathrm{Tr}F^4 = \frac{1}{100}\mathrm{Tr}\left(F^2\right)^2$. This implies that that the only degree eight class comes in the form of a composite, $a_1 \cup a_2$ for a_1 and a_2, the generators of $H^4(X, \mathbb{Z})$ pulled back from $H^4(BE_8, \mathbb{Z})$.

8. Realizing the anomalies in this approach

Given an action S in Euclidean signature, it often splits into a real and an imaginary parts, $S = \mathrm{Re}\,S + \mathrm{i}\,\mathrm{Im}\,S$, so that when forming the semi-classical partition function $\int_{\mathcal{M}} \mathrm{e}^{2\pi \mathrm{i} S}$ one gets a modulus and a phase. The latter is usually given by the topological (i.e. the metric-independent) parts S_{top} of the action as Phase $= \mathrm{e}^{2\pi \mathrm{i}\,\mathrm{Re}\,S} = \mathrm{e}^{2\pi \mathrm{i} S_{top}}$.

[6] One way is to set this to zero rationally by requiring p_2 to be equal to $\frac{7}{4}p_1^2$, but this does not seem to be the best possible.

[7] We thank Michael Hopkins and Isadore Singer for pointing out to us that from a topological point of view, such a structure also only works in low degrees, interestingly in the range of dimension relevant to M-theory.

In studying the topological aspects of the partition function in M-theory, and upon including torsion fields, this phase leads to subtle signs that give potential anomalies. In [2] the condition on the phase ended up being that it is essentially identically one. That involved the study of the divisibility properties of the fields. Since this lives in $\mathbb{Z}_2$, the phase was just given by the mod 2 reduction of the action, which by Witten's earlier result [1] is just the sum of the mod 2 index of the Dirac operator coupled to an E_8 bundle and the mod index of the Rarita–Schwinger operator, i.e. the Dirac operator coupled to the tangent bundle (minus 3 copies of the trivial line bundle). Explicitly [1] [2]

$$\Phi = \exp 2\pi \mathrm{i} \left[\frac{1}{2}\mathrm{Index}(D_{E_8}) + \frac{1}{4}\mathrm{Index}(D_{R.S.})\right]. \tag{8.1}$$

Using the Atiyah–Patodi–Singer index theorem and using the fact that the mod 2 index of the Dirac operator coupled to a real bundle in ten dimensions is a topological invariant, the phase was shown by Witten to reduce to $\Phi = (-1)^{f(a)}$, where $f(a)$ is the mod 2 index of the Dirac operator coupled to the E_8 vector bundle with a degree four class a. In [2] this mod 2 index was studied via torsion pairings on cohomology. On X^{10} and two degree four classes $a, b \in H^4(X^{10}; \mathbb{Z})$, the torsion pairing used is $T(a, Sq^3 b) = \int_{X^{10}} a \cup Sq^2 b$, where by Adem relation, $\beta(Sq^2 b) = Sq^3 b$. In general T takes values in $U(1)$ but in this case it takes values in $\mathbb{Z}_2 \subset U(1)$ since $Sq^3 b$ is 2-torsion. The mod 2 index $f(a)$ is a quadratic refinement of the bilinear form via the cup-product [2]

$$f(a_1 + a_2) = f(a_1) + f(a_2) + \int_{X^{10}} a_1 \cup Sq^2 a_2. \tag{8.2}$$

First, note that we have written I_8 in terms of the Spin characteristic classes. In particular, the expression (2.6) for I_8 includes Q_2, so in order to look at a possible mod 2 reduction of I_8 we need to see what the corresponding reduction of the Q_i's is. The mod 2 reduction r_2 of the Spin classes are the Stiefel–Whitney classes in that dimension, i.e.

$$\begin{aligned} \rho_2(Q_1) &= w_4 \\ \rho_2(Q_2) &= w_8. \end{aligned} \tag{8.3}$$

However, we see that we have the division by 24 which makes the task nontrivial.[8]

The presence of the one-loop term in M-theory $\int_{Y^{11}} C_3 \wedge I_8$ reduced in type IIA string theory to the corresponding one-loop term $\int_{X^{10}} B_2 \wedge I_8$. Similarly, the Chern–Simons term $\frac{1}{6}\int_{Y^{11}} C_3 \wedge G_4 \wedge G_4$ reduces to the corresponding Chern–Simons term in type IIA $\frac{1}{6}\int_{X^{10}} B_2 \wedge F_4 \wedge F_4$. The field F_4 is obtained from the M-theory field G_4 and so is expected to also have a shift proportional to Q_1, the mod 2 reduction of which is w_4. Now the mod 2 reduction of the action amounts to replacing the fields by their mod 2 reductions, together with the mod 2 Steenrod operations,[9] so schematically F_4 should correspond to w_4 and Sq^4, I_8 to w_8, and B_2 to Sq^2. Now we take B_2 to correspond to a cohomology operation given by the second Steenrod Square Sq^2 (that is how it shows up in KO-theory), and so the operation replacing the one-loop term is $\int_{X^{10}} Sq^2 I_8$. By using (2.6) we see that the condition is[10] $Sq^2 Q_2 = 0$. Thus from the topological action we get three possible terms in the mod 2 reduction, namely $w_4 Sq^2 w_4$, $Sq^2 Sq^4 w_4$, and $sq^2 w_8$. In what follows we will show that such terms correspond naturally to expressions in Spin K-theory (see (8.12)). The dimensions relevant here are: four for the M2-brane theory, eight for the M5-brane theory, ten for type II string theory, and twelve for M-theory (more precisely, the cobounding theory).

8.1. The five-brane and eight-manifolds

The topological part of the M5-brane action extended via the Chern–Simons construction from six dimensions to eight dimensions is given by [8,9]

$$S_8 = \frac{1}{2}\int_{M^8} G_4 \wedge G_4 - \lambda \wedge G_4. \tag{8.4}$$

[8] One might be able to evade this subtlety by looking at the integral of the one-loop term (2.4) lifted as usual to a twelve-dimensional bounding Spin manifold Z^{12}. If we assume that the class of G_4 is divisible by 24 then we can write that integral as $\int_{Z^{12}} \frac{G_4}{24} \wedge Q_2$, assuming that G_4 is in cohomology.

[9] We could have included w_2 with B_2, but we are assuming our ten-manifold to be spin.

[10] This involves mod 2 reduction implicitly.

The mod 2 reduction of this action is

$$\rho_2(a) \cup \rho_2(a) - U_2^1 \cup \rho_2(a), \tag{8.5}$$

where we denote by $\rho_2(a)$ the mod 2 reduction of the integral class a of G_4, and U_2^1 is the second Wu class[11] given in terms of the Stiefel–Whitney classes by the Wu formula $U_2^1 = w_4 - w_2^2$. For M^8 spin, which is what we assume, then U_2^1 is the same as w_4. Similarly, the mod 3 reduction takes the form

$$\rho_3(a) \cup \rho_3(a) - U_3^1 \cup \rho_3(a), \tag{8.6}$$

where $\rho_3(a)$ denotes the mod 3 reduction of the integral class a, and U_3^1 is the first Wu class at the prime $p = 3$.

Consider the exact sequence [10]

$$0 \longrightarrow \ker Q_1 \longrightarrow \widetilde{K\mathrm{Spin}}(M^8) \xrightarrow{Q_1} H^4(M^8; \mathbb{Z}) \longrightarrow 0. \tag{8.7}$$

Since the kernel of Q_1 is string manifold, then we see that the difference between this Spin K-theory and integral four cohomology is the string condition. The Spin K-theory picks degree four classes that are in the image of Q_1 modulo the ones in its kernel. Since this looks like cohomology then it makes sense to expect to be able to replace Q_1 by some cohomology operation that would appear in the corresponding Atiyah–Hirzebruch spectral sequence. We further ask the question: what is the meaning of Q_2 once Q_1 vanishes, i.e. for String manifolds? The existence of the exact sequence, which is an isomorphism, [10]

$$Q_2\,|_{\ker Q_1} : \ker Q_1 \longrightarrow 3H^8(M^8; \mathbb{Z}) \tag{8.8}$$

means that once Q_1 is zero, Q_2 coincides with three times the eighth integral cohomology of the manifold. Since in this case Q_2 would be just twice the second Pontrjagin class, $2p_2$, then this implies that p_2 is equal to six times the integral generator. Thus we see that for a String manifold, the second Pontrjagin class is divisible by six. This is obviously consistent with the divisibility by two in the proposal in [5].

8.2. *The mod 2 anomaly*

The discussion leading to the mod 2 reduction of the action involved only the E_8 classes and did not include the gravitational class $\lambda/2$ appearing in the shifted quantization condition for the M-theory four-form (1.1). In particular, they involved the Wu relations among the Chern classes of the unitary bundle obtained from the breaking $E_8 \supset (SU(5) \times SU(5))/\mathbb{Z}_5$ [2]. In our present context of Spin characteristic classes, we would like to give the corresponding condition on these classes. Since $\lambda/2$ appears linearly with a, the Spin classes will have an analogous expression[12] $Q_1 \cup Sq^2 Q_1$. We would like to investigate whether this can be obtained in a systematic way as part of an expression in KSpin which would also have a topological interpretation. In a given dimension, there are relations between the characteristic classes and the cohomology operations. In this case, the relations in $H^{10}(BSO; \mathbb{Z}_2)$ are given as linear combinations of the possible Steenrod square operations acting on the generators (8.3), namely $w_2^2 \cup Sq^2 w_2^2$, $Sq^4 Sq^2 w_2^2$, and $Sq^2 w_4^2$. In the spin case, only the latter survives.

We are dealing with degree four and degree eight classes so we can pull back the above classes to the classifying spaces $K(\mathbb{Z}, 4)$ and $K(\mathbb{Z}, 8)$, since cohomology groups of X can be understood as the homotopy classes of maps from that space to the Eilenberg–Maclane spaces

$$\begin{aligned} H^4(X, \mathbb{Z}) \times H^8(X, \mathbb{Z}) &= [X, K(\mathbb{Z}, 4)] \times [X, K(\mathbb{Z}, 8)] \\ &= [X, K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 8)]. \end{aligned} \tag{8.9}$$

Let $x \in H^4(K(\mathbb{Z}, 4), \mathbb{Z})$ and $y \in H^8(K(\mathbb{Z}, 8), \mathbb{Z})$ be the standard generators. We are further interested in classes in $\mathbb{Z}_2$, so let the corresponding mod 2 reductions be given by

$$\begin{aligned} z_4 &= x \bmod 2 \in H^4(K(\mathbb{Z}, 4), \mathbb{Z}_2) \\ z_8 &= y \bmod 2 \in H^8(K(\mathbb{Z}, 8), \mathbb{Z}_2). \end{aligned} \tag{8.10}$$

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[11] In this general notation, U_2^1 corresponds to v_1 or ν_1 in the notation more particular to the prime 2.

[12] Again the mod 2 reduction is understood implicitly.

In the Postnikov tower with lowest level E_0, $H^{10}(E_0; \mathbb{Z}_2)$ as a vector space over $\mathbb{Z}_2$ has a basis $z_4 \cup Sq^2 z_4$, $Sq^4 Sq^2 z_4$, and $Sq^8 z_8$. It turns out that the coefficients in the linear combination are one so that the second k-invariant is given by [10] $k_2 = z_4 \cup Sq^2 z_4 + Sq^4 Sq^2 z_4 + Sq^8 z_8$, and the corresponding map

$$\Lambda_X : H^4(X; \mathbb{Z}) \times H^8(X; \mathbb{Z}) \to H^{10}(X; \mathbb{Z}_2), \tag{8.11}$$

is given by [10]

$$\Lambda_X(Q_1, Q_2) = Q_1 \cup Sq^2 Q_1 + Sq^4 Sq^2 Q_1 + Sq^2 Q_2. \tag{8.12}$$

We view this map as the mod 2 index for Dirac operators coupled to Spin bundles, and the vanishing of the mod index is then essentially[13] the condition to lift the degree four (and eight) cohomology to Spin K-theory.

8.3. The DFM anomaly

In this section we look at the DFM anomaly [4]. We aim at achieving two things: First, encode the structure of the degree four and degree eight classes in our context of Spin characteristic classes, and second, seek a possible variant of this anomaly to include mod 3 reductions of fields. The first was considered in Section 6.3, so here we consider the second.

In order to describe the electric charge induced by the self-interactions of the C-field, Ref. [4] defined an integral lift of the EOM of G_4, $\Theta_X(a)$, where a is the integral class appearing in the shifted quantization condition of G_4 (1.1). We note that the quadratic refinement is exactly the addition law on the target of the map Q_X in (7.8). Thus we see that the product of the two cohomology groups H^4 and H^8 together with their ring structure encodes the elements a and $\Theta_X(a)$ together with the correct addition laws. Now that we have seen that we have the correct structure for the elements and their addition law, we would like to see what consequence that has on the anomaly itself.

Let us first motivate the problem heuristically from the point of view of ten-dimensional type IIA. There, the Freed–Witten anomaly reads [26] $Sq^3 F + H_3 \cup F = 0$, where F is the total Ramond–Ramond field strength that includes the fields of all even degrees. Since the 'operator' $Sq^3 + H_3\cup$ appearing in this equation is of a uniform degree, we can isolate one of the RR fields. We thus focus on F_4, in which case $Sq^3 F_4 + H_3 \cup F_4 = 0$. We use this expression to get hints about what a possible 'S^1-lift' might be in M-theory. Since the diagonal lift of H_3 as well as the vertical lift of F_4 to M-theory both give G_4, a candidate expression in M-theory would involve replacing F_4 and H_3 both with G_4, i.e. schematically

$$\mathcal{O} G_4 + G_4 \cup G_4, \tag{8.13}$$

where $\mathcal{O}$ is a cohomology operation, we have been arguing for its existence and which need to be determined. Again, in order to get an equation of homogeneous degree – that is the only choice that seems to be available – the operation $\mathcal{O}$ should be of degree four, i.e. it should raise the cohomology degree by four. What are the candidates? It seems to be only Sq^4 (and decomposables) at $p = 2$ or P_3^1 at $p = 3$.

We would like to understand the cohomology groups $H^{4i}(X, \mathbb{Z})$ for $i = 1, 2$ in order to understand the map from Spin K-theory and the corresponding obstructions to lifting. We follow [10], for the mathematical results, for what follows. Given the universal Spin characteristic classes $Q_i \in H^{4i}(B\mathrm{Spin}; \mathbb{Z}) = [B\mathrm{Spin}, K(\mathbb{Z}, 4i)]$, we can pull them back to the space X. To understand the image of Q_X we ask which map $f : X \longrightarrow K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 8)$ admits a lifting relative to the pair $Q = (Q_1, Q_2)$,

$$B\mathrm{Spin} \xrightarrow{\Delta} B\mathrm{Spin} \times B\mathrm{Spin} \xrightarrow{Q_1 \times Q_2} K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 8). \tag{8.14}$$

It is here that the Steenrod power operations P_3^1, taking $H^4(K(\mathbb{Z}, 4); \mathbb{Z}_3)$ to $H^8(K(\mathbb{Z}, 4); \mathbb{Z}_3)$, make their appearance as follows. Let x_4 and y_8 be the standard generators of $H^4(K(\mathbb{Z}, 4); \mathbb{Z})$ and $H^8(K(\mathbb{Z}, 8); \mathbb{Z})$, respectively. Then as vector spaces over $\mathbb{Z}_3$, $H^8(K(\mathbb{Z}, 8); \mathbb{Z}_3)$ is generated by a single element y_8 mod 3, while $H^8(K(\mathbb{Z}, 4); \mathbb{Z}_3)$ is

[13] Note that there are factors of half involved.

generated by the two elements

$$x_4^2 \bmod 3 \quad \text{(decomposable)}$$
$$P^1(x_4 \bmod 3) \quad \text{(primitive)} \tag{8.15}$$

The invariant k_1 is a cohomology class that lies in

$$H^8(K(\mathbb{Z},4) \times K(\mathbb{Z},8); \mathbb{Z}_3) = H^8(K(\mathbb{Z},4); \mathbb{Z}_3) \oplus H^8(K(\mathbb{Z},8); \mathbb{Z}_3), \tag{8.16}$$

and so its expression is given as a linear combination of the above three $\mathbb{Z}_3$-valued generators. It turns out again that the coefficients are all one so that the map

$$R_X : H^4(X; \mathbb{Z}) \times H^8(X; \mathbb{Z}) \to H^8(X; \mathbb{Z}_3) \tag{8.17}$$

given by

$$R_X(a, b) = (a \cup a + b) \bmod 3 + P_3^1(a \bmod 3), \tag{8.18}$$

is a homomorphism, with the group structure being that on $H^4(X; \mathbb{Z}) \times H^8(X; \mathbb{Z})$.[14]

The lifting condition in dimension eight is the following [10]. The stable classes of Spin bundles over an eight-dimensional closed manifold are in one-to-one correspondence with pairs $(a, b) \in H^4(X; \mathbb{Z}) \times H^8(X; \mathbb{Z})$ satisfying

$$(a \cup a + b) \bmod 3 + U_3^1 \cup (a \bmod 3) = 0, \tag{8.19}$$

where U_3^1 is the corresponding Wu class. It is this formula that we think of as the mod 3 analog of the DFM formula.

9. Further remarks

The integral anomaly: For a torsion class c, $f(a+2c) = f(a) + \int c \cup Sq^2\lambda$ [2]. The absence of the refinement implies that in the torsion pairing between a 4-class and a seven-class that $\beta Sq^2\lambda$ be equal to zero. This is $T(b, Sq^3\lambda) = 0$ giving the W_7 anomaly canceled in [3] via elliptic cohomology. The cohomology ring of BSpin over the integers contains, in addition to the Spin characteristic classes Q_1 and Q_2 of dimensions 4 and 8 respectively, a characteristic class of degree seven. This is the generator of

$$H^7(B\mathrm{Spin}; \mathbb{Z}) = \mathbb{Z}_2, \tag{9.1}$$

which is nothing but the Seventh integral Stiefel–Whitney class W_7, obtained as the Bockstein on the sixth mod 2 Stiefel–Whitney class w_6. This is precisely the anomaly that DMW found [2]. It was canceled in [3] by declaring the spacetime to be orientable with respect to Landweber's elliptic cohomology $E(2)$ or Morava K-theory $K(2)$ (both taken at the prime $p = 2$), a result which was obtained by identifying W_7 as the cohomology class corresponding to an obstruction, i.e. as a differential in the Atiyah–Hirzebruch spectral sequence. From (9.1) it seems that there is another interpretation, namely that the vanishing of W_7 is simply the vanishing of the seventh Spin characteristic class pulled back to spacetime from the universal bundle BSpin. Thus, the DMW anomaly can also be naturally interpreted in this context.

The w_4 anomaly: This anomaly was physically proposed and mathematically derived in [3]. This also shows up in an apparently different context, namely as part of the shift in the quantization of the M-theory field strength [1]. We make a connection between the two. We start with the following observation. If $w_4 = 0$ then the first Spin characteristic class is divisible by two. Since $Q_1 \equiv w_4 \bmod 2$, then $w_4 = 0$ implies that $Q_1 \equiv 0 \bmod 2$, which implies that Q_1 is divisible by two. So there is some (not necessarily unique) class γ such that $2\gamma = Q_1$. This gives an interpretation of the $EO(2)$ condition as giving the shift in Witten's quantization (1.1) to be even. In this case, the membrane path integral can be defined with no ambiguity. Thus, the w_4 condition, when traced back, can be viewed as the condition for an anomaly free membrane partition function. In [3] this was needed to construct the mod 2 part of the generalized

[14] In going from k_1 to R_X we replaced x_4 by a and y_8 by b.

cohomology partition function. Thus, we interpret the construction in [3] as corresponding to the case when the M-theory field strength satisfied a direct quantization condition, i.e. one that is not shifted. Note that W_7 is obtained from w_4 via the Steenrod operation Sq^3. By the Wu formula $w_6 = Sq^2 w_4 + w_2 w_4$, so that for spin bundles one has $W_7 = \beta Sq^2 w_4 = Sq^3 w_4$, where β is the Bockstein map.

Mod 4 reduction: The inclusion $i : \mathbb{Z}_2 \to \mathbb{Z}_4$ induces the mapping $i_* : H^*(X;\mathbb{Z}_2) \to H^*(X;\mathbb{Z}_4)$. For a vector bundle ξ, the reduction mod 4 of the Pontrjagin classes $p_i(\xi)$ can be written in terms of the Stiefel–Whitney classes $w_i(\xi)$ (of various degrees) by using i_* above and the Pontjagin square $\mathcal{P}$. The latter is a cohomology operation from $H^{2k}(X;\mathbb{Z}_2)$ into $H^{4k}(X;\mathbb{Z}_4)$. The mod 4 reduction of the Pontrjagin classes is

$$\begin{aligned}\rho_4 p_1(\xi) &= \mathcal{P}w_2(\xi) + i_* w_4(\xi),\\ \rho_4 p_2(\xi) &= \mathcal{P}w_4(\xi) + i_*\{w_8(\xi) + w_2(\xi)w_6(\xi)\}.\end{aligned} \tag{9.2}$$

Thus the mod 4 reductions are given essentially by the mod 2 reductions. Note that for a Spin bundle, $w_2(\xi)$ is zero, and requiring further the $EO(2)$ orientation condition $w_4 = 0$ [3,21] then implies that the mod 4 reduction of p_1 is zero. This would also be true for p_2 if in addition we require w_8 to be zero, i.e. that the second Spin characteristic class Q_2 used earlier is even.

Mod 5 reduction: From the definition of the Steenrod reduced powers we see that the operation P_5^1 cubes a degree four class. Thus, on the mod 5 reduction $\rho_5(G_4)$ we have $P_5^1(\rho_5(G_4)) = \rho_5(G_4) \cup \rho_5(G_4) \cup \rho_5(G_4)$, thus generating the form of the cubic Chern–Simons term. What about the reduction of I_8 mod 5? If we assume for simplicity that $p_1/2 = 0$, then I_8 reduces to $p_2/48$, the mod 5 reduction of which we write as $p_2/2$ mod 120. The Pontrjagin classes mod 120 are topological invariant [15]. If we use Spin bundles and their higher connected analogs then the right classes to look at are the Spin characteristic classes formed of $p_1/2$ and $p_2/2$. We expect that using these classes we get the topological invariance of I_8 reduced modulo 5.

Type II and the AHSS: In type IIA string theory it was argued in [25] that a D-brane which is free of Freed–Witten anomalies lifts to twisted K-theory if and only if the Poincaré dual of the cycle that it wraps is annihilated by the Milnor primitive $Q_1 = -\beta P_3^1$. This operator is indeed the fifth differential d_5 in the Atiyah–Hirzebruch Spectral Sequence for complex K-theory at $q = 3$. Mathematically, this follows from [27] where the differentials at prime $q \geq 2$ are given by $d_{2r(q-1)+1} = \beta P_q^r$. For $q = 3$ we see that the first differential is just the Bockstein β and the third is $d_9 = \beta P_3^2$. This shows that the only nontrivial operation at $q = 3$ in string theory is βP_3^1 considered in [25]. In the light of this discussion, there does not seem to be anything special about $q = 3$ in the considerations in [25] except providing examples and staying within the allowed range of dimension. This suggests that $q = 5$ examples should be relevant in type II but they have to be restricted to degree one classes, as seen by the fact that P_5^1 raises the cohomology degree by eight.

In our current M-theory context, the formula (6.9) suggests an obstruction in a spectral sequence for which we argued earlier. The differential has order four. Even differentials are usually associated with real (rather than complex) theories – for example, whereas the first differential for K-theory is $d_3 = \beta Sq^2$, for *KO*-theory it is $d_2 = Sq^2$, and this generalizes to other theories as well – and so this is compatible with the requirement that the theory be real.

In closing we point out that a further study of denominator factors is needed. This may require going beyond KSpin. We expect, in line of previous work, that accounting for factors such as 24 will make contact with higher $BO\langle n\rangle$. This will be the subject of the next step in our investigation.

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